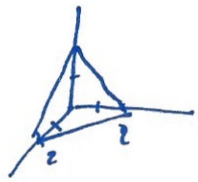
Math 251
Exam IV v2
Spring 2016

Show All Work

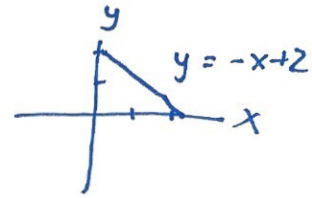
name____________________

1.) **Set up** $\iiint_E x^2 dV$ where E is the solid tetrahedron with vertices (0, 0, 0), (2, 0, 0), (0, 2, 0), and (0, 0, 2)



$$x + y + z = 2$$
$$z = 2 - x - y$$

base



$$\int_0^2 \int_0^{-x+2} \int_0^{2-x-y} x^2 \, dz \, dy \, dx$$

2.) Find $\iiint_E 2z dV$ where E is the region in the first octant below the plane $z + x = 3$ and inside the cylinder $x^2 + y^2 = 1$.

$$\int_0^{\pi/2} \int_0^1 \int_0^{3-r\cos\theta} 2z \, dz \cdot r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 z^2 \Big|_0^{3-r\cos\theta} r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 (3 - r\cos\theta)^2 \cdot r \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 (9 - 6r\cos\theta + r^2\cos^2\theta) \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 (9r - 6r^2\cos\theta + r^3\cos^2(\theta)) \frac{dr}{d\theta} = \int_0^{\pi/2} \frac{9}{2} r^2 - 2r^3\cos\theta + \frac{r^4}{4}\cos^2\theta \Big|_0^1 d\theta$$

$$= \int_0^{\pi/2} \left(\frac{9}{2} - 2\cos\theta + \frac{1}{4}\cos^2\theta\right) d\theta = \int_0^{\pi/2} \left(\frac{9}{2} - 2\cos\theta\right) d\theta + \frac{1}{4}\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{9}{2}\theta - 2\sin\theta \Big|_0^{\pi/2} + \frac{1}{8}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}$$

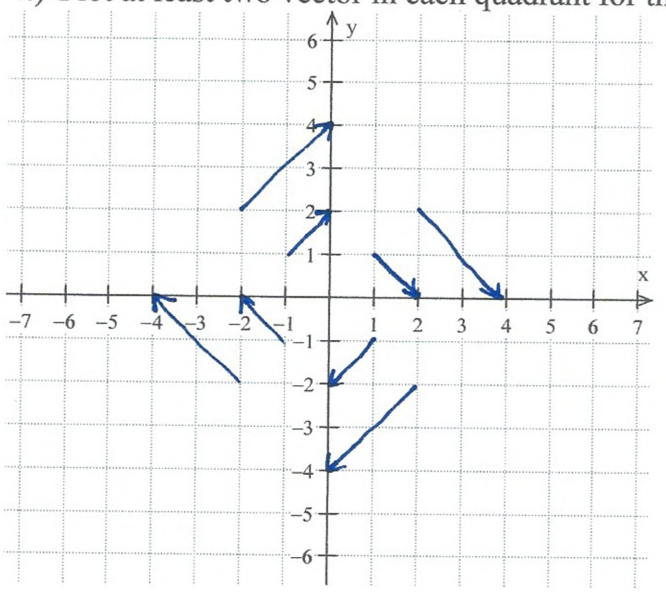
$$= \left(\frac{9\pi}{4} - 2\right) - (0 - 0) + \frac{1}{8}\left(\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{9\pi}{4} - 2 + \frac{\pi}{16}$$

$$= \frac{37\pi}{16} - 2$$

3.) **Set up** $\iiint_H z\sqrt{x^2+y^2}\,dV$ where H is the solid hemisphere that lies **below** the xy-plane and has center origin and radius 2.

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{2} (\rho\cos\varphi)(\rho\sin\varphi)\cdot \rho^2\sin\varphi \; d\rho\, d\varphi\, d\theta$$

4.) Plot at least two vector in each quadrant for the vector field **F** = < y, - x >



point	$F(x\,y)$
$(1,1)$	$\langle 1,-1\rangle$
$(2,2)$	$\langle 2,-2\rangle$
$(1,-1)$	$\langle -1,-1\rangle$
$(2,-2)$	$\langle -2,-2\rangle$
$(-1,-1)$	$\langle -1,1\rangle$
$(-2,-2)$	$\langle -2,2\rangle$
$(-1,1)$	$\langle 1,1\rangle$
$(-2,2)$	$\langle 2,2\rangle$

5.) Evaluate the following line integrals.

a) $\int_C -y\,dx + xy\,dy$ where C is the line segment from (1, 2) to (-1, 6).

$r(t) = \langle 1-2t, 2+4t\rangle \quad 0 \le t \le 1 \qquad r'(t) = \langle -2, 4\rangle$

$$\int_0^1 \left[-(2+4t)(-2) + (1-2t)(2+4t)\cdot 4\right] dt$$

$$= \int_0^1 \left(4+8t + 4(2-8t^2)\right) dt = \int_0^1 (4+8t+8-32t^2)\, dt$$

$$= \int_0^1 (-32t^2+8t+12)\, dt = -\frac{32t^3}{3} + 4t^2 + 12t \Big|_0^1$$

$$= -\frac{32}{3} + 4 + 12 = -\frac{32}{3} + \frac{12}{3} + \frac{36}{3} = \frac{16}{3}$$

b) $\int_C x^2 ds$ where C is the part of the circle $x^2 + y^2 = 4$ that lies in the first quadrant.

$r(t) = \langle 2\cos t, 2\sin t \rangle \qquad 0 \le t \le \pi/2$

$r'(t) = \langle -2\sin t, 2\cos t \rangle$

$$= \int_0^{\pi/2} 4\cos^2 t \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/2} 4\cos^2 t \sqrt{4\sin^2 t + 4\cos^2 t}\, dt$$

$$= \int_0^{\pi/2} 4\cos^2 t \cdot \sqrt{4}\, dt = 8\int_0^{\pi/2} \cos^2 t\, dt = 8\int_0^{\pi/2} \frac{1+\cos 2t}{2}\, dt$$

$$= 4\int_0^{\pi/2} (1+\cos 2t)\, dt = 4\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{\pi/2} = 4\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = 2\pi$$

6.) A)Determine whether the function $\mathbf{F}(x, y) = \langle e^y + 3x^2, xe^y + \cos y \rangle$ is conservative. If it is, find its potential function.

$\frac{dQ}{dx} = e^y = \frac{dP}{dy}$ yes its conservative

Potential function $f(x,y) = xe^y + x^3 + \sin y + K$

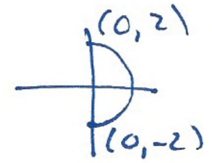
7.) Let F(x, y) = <P(x, y), Q(x, y)>

a) Under what conditions is $\int_C \boldsymbol{F} \cdot \boldsymbol{dr} = \int_C P(x,y)dx + Q(x,y)dy$ independent of path?

Its conservative $\left(\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \right)$

8.) Find the work done by **F**(x, y) = <x² + y², 2xy> where C is traced counter clockwise along the right side of the circle centered at the origin with radius 2.

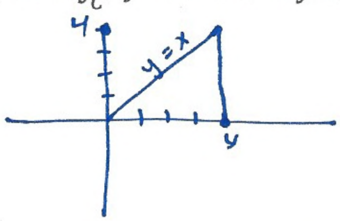
notice $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 2y$ conservative.



$$\int_C \vec{F} \cdot d\vec{r} = \frac{x^3}{3} + xy^2 \Big|_{(0,-2)}^{(0,2)}$$

$$= (0+0) - (0+0) = 0$$

9.) Find $\int_C y^3\,dx + e^x dy$ where C is the triangle from (0,0) to (4,0) to (4, 4) to (0, 0)



closed - GREEN'S THEOREM

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^4 \int_0^x (e^x - 3y^2)\,dy\,dx$$

$$= \int_0^4 ye^x - y^3 \Big|_{y=0}^{y=x} dx = \int_0^4 xe^x - x^3\,dx = \int_0^4 xe^x dx - \int_0^4 x^3 dx$$

1st integral by parts $u = x$, $du = dx$, $dv = e^x$, $v = e^x$

$$= \left[xe^x \Big|_0^4 - \int_0^4 e^x\,dx \right] - \frac{x^4}{4} \Big|_0^4 = xe^x - e^x - \frac{x^4}{4} \Big|_0^4$$

$$= (4e^4 - e^4 - 64) - (0 - 1 - 0) = 3e^4 - 64 + 1$$

$$= 3e^4 - 63$$